Existence of three-dimensional ideal-MHD equilibria with current sheets

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We consider the linear and nonlinear ideal plasma response to a boundary perturbation in a screw pinch. We demonstrate that three-dimensional, ideal-MHD equilibria with continuously-nested flux-surfaces and with discontinuous rotational-transform across the resonant rational-surfaces are well defined and can be computed both perturbatively and using fully-nonlinear equilibrium calculations. This rescues the possibility of constructing MHD equilibria with current sheets and continuous, smooth pressure profiles. The results are also of direct practical importance since they predict that, even if the plasma acts as a perfectly conducting fluid, a resonant magnetic perturbation can penetrate all the way into the center of a tokamak without being shielded at the resonant surface.

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Introduction. The properties and numerical computation of three-dimensional, ideal-MHD equilibria is of fundamental importance for understanding the behaviour of both magnetically confined fusion and astrophysical plasmas [1–3]. In particular, singular current densities at rational surfaces are predicted in equilibria with continuously-nested flux-surfaces, with and without pressure [1, 4–7]. These singular currents play a crucial role in describing (i) the plasma response to non-axisymmetric boundary perturbations [8–10], (ii) ideal and resistive stability [11, 12], and (iii) the dynamics of reconnection phenomena, such as sawteeth [13].

The singularities arise from requiring charge conservation, $\nabla \cdot \mathbf{j} = 0$, which gives rise to a magnetic differential equation for the parallel current, $\mathbf{B} \cdot \nabla u = -\nabla \cdot \mathbf{j}_\perp$, where $\mathbf{j} \equiv u\mathbf{B} + \mathbf{j}_\perp$. Magnetic differential equations are densely singular [14]. Their singular nature is exposed in straight-field-line coordinates, (ψ, θ, ζ), which may be constructed on each flux surface and imply $\mathcal{J}\mathbf{B} \cdot \nabla = \iota\!\!\!-\, \partial_\theta + \partial_\zeta$, where $\mathcal{J}$ is the Jacobian of the coordinates, $\iota\!\!\!-(\psi)$ is the rotational-transform on a given flux surface, which is labeled by the enclosed toroidal flux, ψ, and θ and ζ are poloidal and toroidal angles. Writing $u = \sum_{m,n} u_{mn}(\psi) \exp\left[i(m\theta - n\zeta)\right]$, we have

$$u_{mn}(x) = h_{mn}(x)/x + \Delta_{mn}\delta(x) \qquad (1)$$

where $x \equiv \iota\!\!\!- m - n$, $h_{mn} \equiv i(\mathcal{J}\nabla\cdot\mathbf{j}_\perp)_{mn}$, and Δ_{mn} is an as-yet undetermined constant. The force-balance equation in its simplest form, $\mathbf{j}\times\mathbf{B} = \nabla p$, implies $\mathbf{j}_\perp = \mathbf{B}\times\nabla p/B^2$; and thus h_{mn} is proportional to the pressure-gradient, $h_{mn} \sim p'$; although it could potentially vanish if if $\mathbf{B}$ were tailored so as to make each $(\mathcal{J}\nabla \cdot \mathbf{j}_\perp)_{mn}$ vanish on its respective resonant surface [15].

The singularities consist of a pressure-driven, Pfirsch-Schlüter $1/x$ current density that arises *around* rational surfaces, and a Dirac δ-function current density that develops *at* rational surfaces as a necessary mechanism to prevent the formation of islands that would otherwise develop in a non-ideal plasma [16]. Only recently have these singular current densities been computed numerically [17]. In doing so, it was realized that infinite shear at the rational surfaces was required in order to have well-defined solutions [17]. This crucial observation is explored in this letter, and it leads to some remarkable conclusions.

Singularities in the current *density* are allowed in the ideal-MHD model, but the total current, $\int_{\mathcal{S}} \mathbf{j} \cdot d\mathbf{s}$, passing through any surface, $\mathcal{S}$, must remain finite for any physically acceptable equilibrium. While the integral of a δ-current density is always finite, surfaces may be constructed through which the Pfirsch-Schlüter current diverges logarithmically. This is not physical: it would seem that the ideal-MHD equilibrium model has the fatal flaw of not allowing for pressure.

However, as noted by Grad [1], equilibrium solutions without infinite currents may be constructed by considering pressure profiles that are flat in a small neighbourhood of each rational surface. In order to construct non-trivial, *continuous* pressure profiles, the pressure-gradient must be finite on a set of finite measure, e.g. the irrationals that are sufficiently far from low order rationals, i.e. those that satisfy a Diophantine condition [18]. In that case, p is continuous but its derivative is not. The pressure profile must be fractal, with the pressure-gradient being discontinuous on a fractal set of finite measure. Grad [1] described such equilibria as pathological.

Alternatively, Bruno and Laurence [19] showed that equilibria with *discontinuous* pressure profiles are also possible solutions, with the finite set of discontinuities in p occurring on surfaces with *strongly-irrational* transform. These "stepped-pressure" states are extrema of the multi-region, relaxed energy functional [20] and resolve the singularities by allowing local relaxation, and thus are not globally ideal.

The question remains: are there well-defined, non-pathological, globally-ideal, MHD equilibrium solutions in arbitrary, three-dimensional geometry, with continuously nested surfaces and with arbitrary, smooth, continuous pressure profiles? In this letter, we will suggest a new class of solution that satisfies each of these conditions.

Historically, the cause of pathologies in MHD equilibria with nested flux-surfaces has been attributed to the class of possible pressure profiles. However, the form of the pressure profile – whether it be smooth, continuous, or pathological in some sense – is not the cause of the problem. The problem is the existence of singularities in the magnetic differential equation, and these singularities are associated with the existence of flux-surfaces with rational rotational-transform. In fact, the singularities remain even for zero pressure. As we shall see, ideal-MHD equilibria with rational surfaces are not analytic functions of the boundary, and it is only by removing the rational surfaces that the singularities can be eliminated.

Cylindrical equilibria with singular currents. Our approach is valid in arbitrary geometry, but for a transparent presentation and to enable verification calculations we consider the linear and nonlinear, ideal plasma response to a non-axisymmetric boundary perturbation in a screw pinch with zero pressure and no flow. The linear plasma displacement, $\boldsymbol{\xi} = \xi^r \mathbf{e}_r + \xi^\theta \mathbf{e}_\theta + \xi^z \mathbf{e}_z$, induced by a non-axisymmetric, radial perturbation with a single Fourier harmonic, $\xi_a \cos(m\theta + kz)$, to the boundary satisfies the linearized force-balance equation, $L_0[\boldsymbol{\xi}] = 0$, where $L_0[\boldsymbol{\xi}] \equiv \delta\mathbf{j}[\boldsymbol{\xi}] \times \mathbf{B} + \mathbf{j} \times \delta\mathbf{B}[\boldsymbol{\xi}]$, where the 'ideal' linear perturbation to the magnetic field is $\delta\mathbf{B}[\boldsymbol{\xi}] \equiv \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$, and $\delta\mathbf{j}[\boldsymbol{\xi}] \equiv \nabla \times \delta\mathbf{B}[\boldsymbol{\xi}]$. This reduces to Newcomb's equation [21],

$$\frac{d}{dr}\left(f \frac{d\xi}{dr}\right) - g\xi = 0, \tag{2}$$

where $\xi^r \equiv \xi(r)\cos(m\theta + kz)$. The functions $f(r)$ and $g(r)$ are determined by the equilibrium,

$$\begin{aligned} f &= B_z^2(\iota\!\!\!- - \iota\!\!\!-_s)^2 \frac{r^3}{R^2 + r^2 \iota\!\!\!-_s^2}, \\ g &= B_z^2\Big[(\iota\!\!\!- - \iota\!\!\!-_s)^2(k^2r^2 + m^2 - 1)\bar{k} + (\iota\!\!\!-_s^2 - \iota\!\!\!-^2)2\bar{k}^2 r\Big], \end{aligned}$$

where $2\pi R$ is the length of the cylinder, $k = -n/R$, $\iota\!\!\!-_s = n/m$, and $\bar{k} = r/(R^2 + r^2\iota\!\!\!-_s^2)$. The equilibrium axial field, $B_z(r)$, satisfies force-balance, $d/dr[B_z^2(1 + \iota\!\!\!-^2 r^2/R^2)] + 2r\iota\!\!\!-^2 B_z^2/R^2 = 0$. The equilibrium is defined by the rotational-transform, $\iota\!\!\!-(r) = RB_\theta/rB_z$, and the major and minor radius, R and a. We choose $\iota\!\!\!-(r) = \iota\!\!\!-_0 - \iota\!\!\!-_1(r/a)^2$ with $\iota\!\!\!-_0 = 0.56$, $\iota\!\!\!-_1 = 0.26$, $a = 0.1$ and $R = 1$, thus placing the rational surface $\iota\!\!\!-_s = 1/2$ at $r_s = a/2$.

Newcomb's equation is singular where $\iota\!\!\!-(r_s) = n/m$. For $m > 1$, and for a continuous $\iota\!\!\!-(r)$ that contains the resonance, $\iota\!\!\!- = \iota\!\!\!-_s$, the solution that is regular at the origin is $\xi(r < r_s) = 0$ and $\xi(r \geq r_s) \neq 0$, i.e. the radial displacement is discontinuous (Figure 1). This class of solutions is obtained by the linearly-perturbed, ideal equilibrium codes that are used to study non-axisymmetric boundary perturbations in tokamaks [3] and stellarators [22]. However, a discontinuous plasma displacement is inconsistent with the assumption of nested flux-surfaces: in fact, magnetic surfaces overlap if the displacement anywhere has $|d\xi/dr| > 1$.

This intuitive condition can be shown from a purely geometrical point of view: if we write the position vector as $\mathbf{x} = (r + \xi^r)\cos\theta\,\hat{\mathbf{i}} + (r + \xi^r)\sin\theta\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, where most generally $\xi^r(r,\theta,z) = \sum_{m,k}\xi_{mk}(r)\cos(m\theta + kz)$, then the Jacobian is $\mathcal{J} = (r + \xi^r)(1 + \partial_r \xi^r)$. For a single harmonic, $\mathcal{J} > 0$ if and only if $|d\xi/dr| < 1$.

The inconsistency of the linear solution originates because ideal-MHD equilibria with resonant surfaces are not analytic functions of the three-dimensional boundary. The linear operator, $L_0[\boldsymbol{\xi}]$, is singular; and higher order terms in the perturbation expansion successively diverge. Writing the perturbation in the geometry as $\boldsymbol{\xi} \equiv \epsilon\,\boldsymbol{\xi}_1 + \epsilon^2\,\boldsymbol{\xi}_2 + \ldots$, the equation for the second order term is $L_0[\boldsymbol{\xi}_2] = -\delta\mathbf{j}[\boldsymbol{\xi}_1] \times \delta\mathbf{B}[\boldsymbol{\xi}_1]$, and $\boldsymbol{\xi}_2$ is even more singular than $\boldsymbol{\xi}_1$.

That perturbation theory is not a valid approach for treating ideal-MHD equilibria has long been known: Rosenbluth *et al.* arrived at a similar conclusion when studying the ideal internal kink [11], stating that "we must abandon the perturbation theory approach and go instead to a boundary layer theory". Rosenbluth *et al.* also realized that "all harmonics are excited to comparable amplitude" by a small boundary deformation.

An equilibrium model that is not an analytic function of the boundary, or one that requires a fractal radial grid to numerically resolve a fractal pressure profile, is decidedly *not* attractive from a numerical perspective. There is, however, a physics-based resolution of these problems that heretofore has not been considered and yet is remarkably simple.

In recent work [17], we computed the singular current densities in ideal-MHD equilibria as predicted by Eq. (1), and we recognized that the non-overlapping of surfaces is ensured by including locally-infinite shear at the resonant surfaces. Extending this idea, we now consider three-dimensional, non-axisymmetric, ideal-MHD equilibria with *discontinuous* rotational-transform.

Revisiting Newcomb's solutions. We reconsider the screw pinch equilibrium, but now with a rotational-transform profile that has a discontinuity at the resonant surface, $\Delta\iota\!\!\!- \equiv \iota\!\!\!-(r_s^+) - \iota\!\!\!-(r_s^-) > 0$. Specifically, the transform is $\iota\!\!\!-(r) = \iota_0 - \iota_1(r/a)^2 \pm \Delta\iota\!\!\!-/2$, where $\pm$ refers to either side of the resonant surface. The discontinuity in $\iota\!\!\!-$ manifests itself in the form of a "DC" current sheet, by which we mean that the average of the current density over the resonant surface is not zero. Figure 1 shows the result of numerical integration of Eq. (2) for different values of $\Delta\iota\!\!\!-$. The linear radial displacement is continuous and smooth provided $\Delta\iota\!\!\!- \neq 0$.

Even for a small, local change in the transform profile, i.e. a small jump $\Delta\iota\!\!\!-$, the global solution is significantly different and the displacement penetrates inside the resonant surface and into the origin. While $\xi(r)$ is continuous and smooth, there is still a jump in the tangential perturbed magnetic field, $\delta\mathbf{B}$, and thus there remains a singularity in the resonant harmonic of the current density, as in the case of continuous transform.

Minimum current sheet. In the limit $\Delta\iota\!\!\!- \to 0$, the

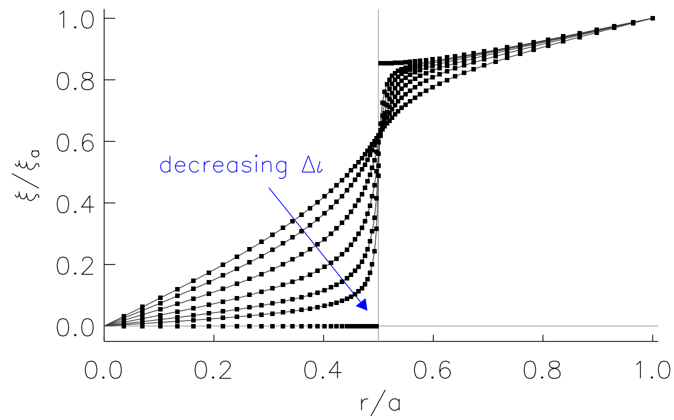

FIG. 1: Solutions of Eq. (2) for an $m = 2$, $n = 1$ boundary perturbation and for different values of $\Delta\iota\!\!\!-$ (solid lines), ranging from $\Delta\iota\!\!\!- = 4 \times 10^{-2}$ to $\Delta\iota\!\!\!- = 10^{-3}$. Also the singular case $\Delta\iota\!\!\!- = 0$ is shown (discontinuous curve). The corresponding SPEC linear calculations are also shown (squares).

linear displacement becomes discontinuous. Thus, there must be a critical value for the magnitude of the DC current sheet below which $|d\xi/dr| > 1$. We estimate it analytically by studying the asymptotic behaviour of $\xi(r)$ around $r = 0$ and $r = r_s$. Expanding $f(r)$ and $g(r)$ around $r = 0$, and using the ansatz $\xi \sim r^{\alpha}$, we see that $\alpha = \pm|m| - 1$. For $m > 1$, the divergent solution must be rejected, and $\xi \sim r^{m-1}$ near the origin. Near the resonant surface we introduce $x = |(\iota\!\!\!- - \iota\!\!\!-_s)/\iota\!\!\!-'_s|$. We assume that $\iota\!\!\!-'$ is constant in the vicinity of the resonant surface, except at $r = r_s$ where it is formally undefined for any $\Delta\iota\!\!\!- \neq 0$. An asymptotic expansion around $x = 0$ with ansatz $\xi \sim x^{\alpha}$ gives $\alpha \in \{-1, 0\}$. The small solution, $\alpha = 0$, must be rejected because it diverges at the origin, and thus $\xi \sim x^{-1}$ around the resonant surface. Using $|\xi'| = |\xi/x|$, we derive an expression for the maximum gradient of the displacement, which happens at the resonant surface, $x = x_s = \Delta\iota\!\!\!-/(2\iota\!\!\!-'_s)$, and is given by

$$|\xi'_s| = 2\iota\!\!\!-'_s \frac{\xi_s}{\Delta\iota\!\!\!-} \qquad (3)$$

for small x_s and where $\xi_s \equiv \xi(r_s)$. Since ξ_s scales with ξ_a, we see that ξ'_s is proportional to the boundary perturbation and inversely proportional to $\Delta\iota\!\!\!-$.

The *sine qua non* condition for the existence of equilibria is $|\xi'| < 1$, which translates into $\Delta\iota\!\!\!- > \Delta\iota\!\!\!-_{min}$, where

$$\Delta\iota\!\!\!-_{min} = 2\iota\!\!\!-'_s \xi_s \ . \qquad (4)$$

The continuous transform limit is recovered as $\iota\!\!\!-'_s \xi_s \to 0$, i.e. for infinitesimally small perturbation or infinitesimally small shear. As a rough estimate we may use $\xi_s \approx \xi_a/2$, and by defining $\epsilon = \xi_a/a$ and $\Delta\iota\!\!\!-_0 = \iota\!\!\!-'_s a$ we have $\Delta\iota\!\!\!-_{min} \approx \epsilon \Delta\iota\!\!\!-_0$, and the minimum DC current sheet scales as $\Delta\iota\!\!\!-_{min} \sim \epsilon$.

This analysis is linear and *a priori* limited to small boundary perturbations, $\epsilon \ll 1$; however, the prediction remains valid for the nonlinear calculations, as we now show.

Nonlinear calculations. Solving for the nonlinear terms analytically, e.g. by inverting $L_0[\boldsymbol{\xi}_2] = -\delta\mathbf{j}[\boldsymbol{\xi}_1] \times \delta\mathbf{B}[\boldsymbol{\xi}_1]$, is now possible because L_0 is non-singular, but this does become a rather cumbersome approach. Instead, we now proceed by using fully self-consistent, nonlinear, numerical calculations, which are valid in arbitrary geometry.

Presently, the widely-used, three-dimensional, nonlinear ideal-MHD equilibrium codes VMEC [23] and NSTAB [24] are restricted to work with smooth functions and cannot compute equilibria with discontinuous rotational-transform. The SPEC code [25] *does* allow for discontinuities. SPEC formally finds extrema of the multi-region, relaxed, MHD (MRxMHD) energy functional, as proposed by Hole, Hudson and Dewar [20, 26]. While in globally-ideal equilibria the topology of the magnetic field is *continuously* constrained, in MRxMHD the topology is *discretely* constrained at a finite number, N, of so-called, "ideal" interfaces, where discontinuities in the pressure and tangential magnetic field are allowed. This was originally intended to accomodate non-trivial pressure profiles in equilibria with partially-relaxed magnetic fields, and MRxMHD equilibria correspond to the stepped-pressure states suggested by Bruno and Laurence [19]; however, MRxMHD has been shown to exactly retrieve ideal MHD in the formal limit $N \to \infty$ [27], and SPEC was recently used [17] to compute the singular current densities in ideal-MHD equilibria as predicted by Eq. (1).

Here we employ SPEC in the "ideal limit", i.e. very large N, to perform linear and nonlinear, ideal equilibrium calculations for the perturbed screw pinch. In this limit the MRxMHD energy functional reduces to $W \equiv \int \left[p/(\gamma - 1) + B^2/2\right] dv$. Equilibrium states are obtained when the gradient of this functional, $F[\mathbf{x}, b] \equiv \nabla p - \mathbf{j} \times \mathbf{B}$, is zero, where $\mathbf{x}$ represents the geometry of the internal flux-surfaces and where b denotes the dependence of the equilibrium on the prescribed boundary.

For verification, we compare linearized, SPEC calculations to the solutions of Newcomb's equation, for both $\Delta\iota\!\!\!- = 0$ and $\Delta\iota\!\!\!- \neq 0$. Given an equilibrium state, i.e. $F[\mathbf{x}, b] = 0$, the first order correction to the internal geometry induced by a boundary deformation, δb, is defined by $\nabla_{\mathbf{x}} F \cdot \boldsymbol{\xi} + \nabla_b F \cdot \delta b = 0$, which is essentially Newcomb's equation generalized to arbitrary geometry, and the solution is $\boldsymbol{\xi} = -(\nabla_{\mathbf{x}} F)^{-1} \cdot \nabla_b F \cdot \delta b$. Figure 1 shows the results of the comparison: the agreement is excellent. The SPEC calculation used $N = 128$ ideal interfaces, which were packed near the resonant surface, so that the ideal-limit is well approximated.

Generally, nonlinear solutions to $F[\mathbf{x}, b] = 0$ for a given boundary are found by iterating on the linear correction, i.e. $\mathbf{x}_{i+1} \equiv \mathbf{x}_i - (\nabla_{\mathbf{x}} F)^{-1} \cdot F$, where i labels iterations. Newton-style methods are particularly efficient, as the corrections converge quadratically; but this is only true if the equations are analytic and this is not the case for ideal equilibria with rational surfaces. SPEC can also em-

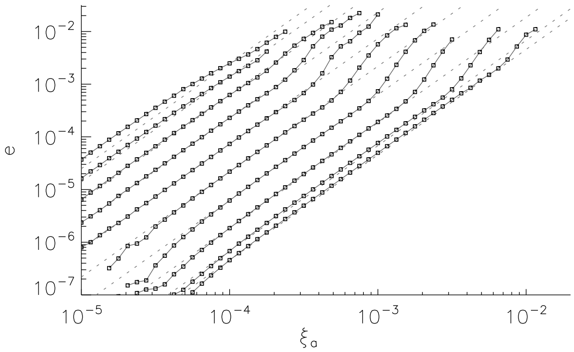

FIG. 2: Convergence of the error between linear and nonlinear SPEC equilibria as the boundary perturbation ξ_a is decreased, and for different values of $\Delta\iota$.

ploy descent-style algorithms similar to that employed by VMEC and NSTAB to minimize the energy functional, e.g. $\partial_\tau \mathbf{x} = -\nabla_{\mathbf{x}} F$, where τ is an arbitrary integration parameter; but these algorithms are similarly adversely impacted by the operator $\nabla_{\mathbf{x}} F$ being singular.

The presence of a discontinuity in ι eliminates the resonant surfaces and allows the nonlinear equilibrium calculations, which a-priori assume smoothly-nested, non-overlapping flux-surfaces, to be precisely compared to the predicted linear displacement, which also gives non-overlapping surfaces provided $\Delta\iota > \Delta\iota_{min}$.

We perform a convergence study of the nonlinear SPEC equilibria towards the corresponding linear prediction as the boundary perturbation ξ_a is decreased and for different values of $\Delta\iota$. Excellent convergence is shown in Figure 2, with the error scaling as $e \sim O(\xi_a^2)$. The nonlinear calculations used Fourier harmonics with $m \leq 6$ and $n \leq 3$. We remark that the agreement arising from this verification exercise is of unprecedented nature and may shed some light on how to reconcile the recently observed discrepancies between equilibrium codes that impose nested surfaces and those that do not [9, 10].

In order to gain insight about the existence of ideal-MHD equilibria with nested flux-surfaces, we now project the nonlinear equilibria onto the parameter space $(\epsilon, \Delta\iota)$ and measure the maximum of the gradient of the displacement (Figure 3). As predicted by Eq. (4), we observe a region where nonlinear equilibria cannot exist because the condition $\max_{r,\theta,z} |d\xi/dr| < 1$ is violated. For example, for a boundary perturbation of $\epsilon \sim 1\%$, a physically-valid ideal equilibrium must have a current sheet with $\Delta\iota \gtrsim 10^{-3}$. Figure 4 shows, for a given perturbation, ϵ, the contributions to $|d\xi/dr|$ from the different harmonics of the displacement. The gradient of the linear term, $\xi'_{2,1}$, dominates that of the higher harmonics, $\xi'_{4,2}$ and $\xi'_{6,3}$, when $\Delta\iota$ is large; and it is only as $\Delta\iota_{min}$ is approached that the nonlinear terms become of comparable amplitude, approaching order one. In fact, as Figure 4 shows, $\xi'_{mn} \sim (\epsilon/\Delta\iota)^n$.

Discussion. We have shown that three-dimensional ideal-MHD equilibria with continuously-nested flux-

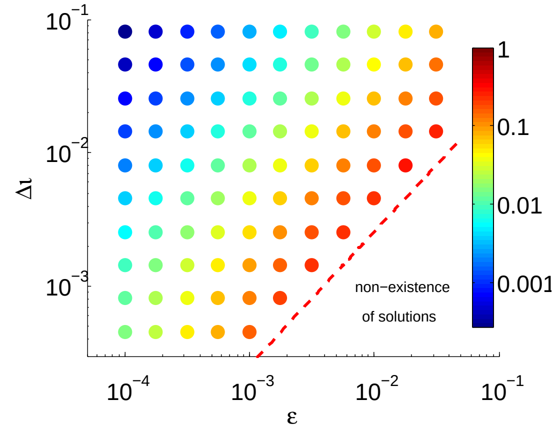

FIG. 3: Existence of nonlinear equilibrium solutions. Each dot represents a nonlinear calculation obtained with SPEC for a given boundary perturbation ϵ and rotational-transform jump $\Delta\iota$. Colormap gives $\max_{r,\theta,z} |d\xi/dr|$. Dashed line is the predicted breaking condition, i.e. Eq. (4).

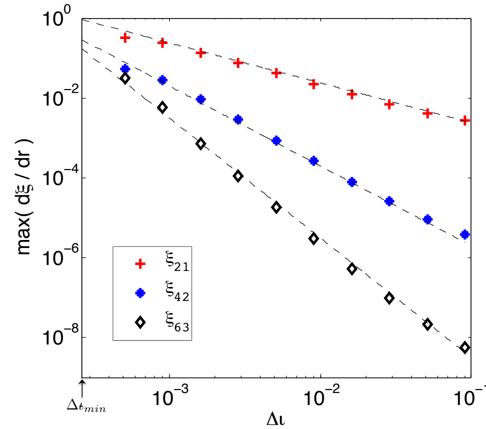

FIG. 4: Contribution from the different harmonics to the maximum gradient of ξ. From SPEC nonlinear simulations for an $m = 2$, $n = 1$ perturbation with $\epsilon = 10^{-3}$. Dashed lines have slope -1, -2, -3. The value of $\Delta\iota_{min}$ is indicated.

surfaces and discontinuous rotational-transform across resonant rational surfaces are well defined and can be computed both perturbatively and using three-dimensional, nonlinear equilibrium calculations. The discontinuity manifests itself in the form of a DC current sheet that ensures the non-overlapping of magnetic surfaces. We have provided a theoretical estimate for the minimum magnitude of this current sheet in a screw pinch, and the predictions have been verified against linear and nonlinear equilibrium calculations.

Our conclusions are general: the generalization of Newcomb's equation to toroidal geometry, as derived by Bineau [28, 29], has the same singular nature as Newcomb's equation, and similar conclusions in toroidal geometry are to be expected.

Our results are of direct practical importance since they predict that, even if the plasma acts as a perfectly conducting fluid, a resonant magnetic perturbation can penetrate all the way into the center of a tokamak without being shielded at the resonant surface; a prediction which has potential implications for using resonant magnetic perturbations for the suppression of edge-localized instabilities, as is planned for ITER [30].

Technically speaking, there are no rational surfaces. While the expected δ-function current densities persist, which is perfectly acceptable in ideal-MHD, the unphysical, infinite, pressure-driven currents are eliminated. In fact, the $1/x$ term in Eq. (1) is bounded by $1/\Delta t_{min}$. This rescues the possibility of constructing 3D MHD equilibria with continuous and smooth pressure profiles.

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